Optimal Trading Strategy of an Informed Trader in the Presence of an Arbitrageur

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1 Background

In the context of a financial market, informed traders are those that possess intelligent information that is not generally available to the public. This information can result in a trading strategy that would presumably provide higher returns than that available to the public. Insider trading has been studied before by [Kyl85], [ST08], [JMZ03], [BCW00]. Kyle [Kyl85] considers the case of a single risk-neutral insider that aims to maximize profit in the presence of noise traders. Back et al. [BCW00] study the optimal trading strategy with more than one informed trader and discuss how the correlation between the traders' information can cause imperfect competition among them. Jeng et al. [JMZ03] evaluate the performance of insider trading in order to estimate their returns.

2 Basic Problem

In this project, we study the case of a single-stock, finite-horizon, informed trader which aims to maximize his expected trading profit in the presence of an arbitrageur. The optimal execution strategy is the policy that maximizes the expected profit and specifies the amounts to be traded on each trading period. Furthermore, we assume that the trader has private information about the price of the stock at a specified future date, T.

We assume that the price of the stock at the end of the horizon, p_T, is higher than the starting price, p_0, $i.e. p_T = kp_0 + p_0$, where $k > 0$. The trader knows the exact value of k and T. We also assume that the trader is forced to liquidate at time T, $i.e. \sum_{k=1}^{T} u_t = 0$.

The stock price at time t, p_t, is unknown. Existing literature suggests models for the price of a stock over time. Here we adopt the model suggested by Bertsimas and Lo [BL98], which takes into account the effect of the traded quantity on future prices. According to this model, $p_t = p_{t-1} + \lambda u_t + \epsilon_t$, where $\epsilon_t \sim \mathcal{N}(0, \sigma_\epsilon^2)$. Since the price is modeled as a stochastic process, the objective would be to maximize the expected profit over the price distribution.

3 Problem Formulation

The problem formulation consists of two parts: formulating trader's actions and objective, and formulating arbitrageur's actions and objective. At each time t, the trader's information set consists of his previous actions, the arbitrageur's actions, the set of past prices, and arbitrageur's estimate of trader's actions and information. The arbitrageur's information set consists of his actions, the set of past prices, and arbitrageur's estimate of trader's actions. Note that the arbitrageur does not know the actions of the trader. The trader does not observe arbitrageur's actions either, but he can repeat the same thought-process as the arbitrageur at each time step and derive arbitrageur's actions as the action that would maximize arbitrageur's objective. Also, note that the trader and the arbitrageur decide on actions u_t and v_t respectively before the price p_t is realized, but the transactions u_t and v_t occur at the realized price p_t. In other words, the arbitrageur and the trader have an estimate of today's closing price, p_t, which is what they base their decisions about u_t and v_t on. Trader's daily cash flow is $-u_t p_t$, and arbitrageur's daily cash flow is $-v_t p_t$.

The arbitrageur's belief about trader's action and information is updated at each time interval after the change in price is observed. Let us denote this belief through a probability distribution $\phi_t(x, k)$, which indicates the probability that the trader's position is x and the trader's private information is k. Assume that the arbitrageur has an initial belief over (x, k), which is denoted by ϕ_0. At each time the arbitrageur observes price changes and estimates what x and k can lead to the corresponding change in price.

Let us denote the policy of the trader by $\{\pi_1, \pi_2, \ldots, \pi_T\}$ and the policy of the arbitrageur by $\{\psi_1, \psi_2, \ldots, \psi_T\}$. The trader's initial information set consists of (p_0, T, k, ϕ_0). Here we adopt the price model used by [BL98] for $t = 1, \ldots, T-1$, *i.e.*,

$$p_t = p_{t-1} + \lambda(u_t + v_t) + \epsilon_t$$

where $\epsilon_t \sim \mathcal{N}(0, \sigma_\epsilon^2)$; we also assume that the price at time T is $p_T = p_0 + kp_0$.

At each time t the trader trades according to the policy function $\pi_t(x_{t-1}, y_{t-1}, \phi_{t-1})$ and the arbitrageur trades according to the policy function $\psi_t(y_{t-1}, \phi_{t-1})$. We also assume that the initial and final positions are zero, *i.e.*, $x_0 = 0$, $x_T = 0$, $y_0 = 0$ and $y_T = 0$.

The trader's and arbitrageur's objective is to maximize their own profit, which is $\sum_{i=1}^{T} x_i \Delta p_{i+1}$, and $\sum_{i=1}^{T} y_i \Delta p_{i+1}$ respectively, and $\Delta p_{i+1} = p_{i+1} - p_i$.

At each time, the arbitrageur updates its belief over (x_t, k) after observing the price. Since the trader knows ϕ_0, the same thought process can be repeated by the trader; therefore the trader can obtain y_t and ϕ_t.

The trader chooses its action u_t from the policy $\pi_t(x_{t-1}, y_{t-1}, \phi_{t-1})$ so as to maximize its value function, which is the expected future profit. This is denoted by $U_t(x_t, y_t, \phi_t)$. Similarly, the arbitrageur chooses its action v_t from the policy $\psi_t(y_{t-1}, \phi_{t-1})$ so as to maximize its value function, which is the expected future profit. This is denoted by $V_t(y_t, \phi_t)$.

4 Belief and Strategy Computation

4.1 Overview

In this section, we restrict our attention to quasi-linear policies and find the perfect Bayesian equilibrium. We divide the problem into four steps:

1. Find a belief update mechanism, through which the arbitrageur's belief over the pair (x_t, k), denoted ϕ_t, is updated after each price observation.

2. Determine a perfect bayesian equilibrium for the trader and the arbitrageur by finding a quasi-linear policy that results in a fixed point.

3. Find a recursive formula for obtaining the value function coefficients, as well as the coefficients of the policy functions.

4. Calculate the value functions by backward recursion using dynamic programming.

In the following sections, we explain how these steps can be accomplished.

4.2 Belief Update

4.2.1 Information Content of x and k

We look at this problem from an information theoretic point of view to compare the information content in x and k and gain a better understanding of the underlying information transmission mechanisms. We consider the trader to be the transmitter, and the arbitrageur to be the receiver. The trader has private information k, and communicates with the market(and the arbitrageur) through his actions x. The arbitrageur is observing the noisy channel and is trying to decode trader's messages to obtain as much information as possible. Therefore, the information that the arbitrageur an obtain by estimating $(x_1, x_2, \ldots, x_t)$ is greater than or equal to the information content that can be extracted by estimating k. This is because the arbitrageur can only connect to the trader by observing the price, which is a noise-corrupted function of x.

However, since estimating $(x_1, x_2, \ldots, x_t)$ requires a growing t-dimensional distribution, we instead consider a smaller information set consisting of (x_t, k) in presence of quasi-linear policies.

4.2.2 Gaussian Two-Dimensional Belief Update under Quasilinear Policies

Assume that arbitrageur's estimate of trader's policy has the form

$$\hat{\pi}_t(x_{t-1}, y_{t-1}, \phi_{t-1}) = \hat{\alpha_{1,t}} x_{t-1} + \hat{\alpha_{2,t}} y_{t-1} + \hat{\alpha_{3,t}} \mu_{x,t-1} + \hat{\alpha_{4,t}} k_{t-1} + \hat{\alpha_{5,t}} k.$$

Define,

$$C_x^t = \left(-\sigma_\epsilon^2 - \lambda^2 \hat{\alpha_{1,t}}^2 \sigma_{x,t-1}^2 + \lambda^2 \hat{\alpha_{1,t}}^2 \sigma_{x,t-1}^2 \rho_{x,k,t-1}^2\right) \sigma_{k,t-1}$$

$$C_k^t = \left(-\sigma_\epsilon^2 - \lambda^2 \hat{\alpha_{5,t}}^2 \sigma_{k,t-1}^2 + \lambda^2 \hat{\alpha_{5,t}}^2 \sigma_{k,t-1}{}^2 \rho_{x,k,t-1}^2\right) \sigma_{x,t-1}$$

$$\theta_x^t = \frac{\lambda^2 \hat{\alpha_{1,t}} \sigma_{x,t-1}^2 \left(\sigma_{k,t-1}\right) \left(1 - \rho_{x,k,t-1}^2\right)}{C_x}$$

$$\theta_k^t = \frac{\lambda^2 \hat{\alpha_{5,t}} \sigma_{k,t-1}^2 \left(\sigma_{x,t-1}\right) \left(1 - \rho_{x,k,t-1}^2\right)}{C_k}$$

Theorem 4.1 *If ϕ_{t-1} is Gaussian, and $\hat{\pi}_t(x_{t-1}, y_{t-1}, \phi_{t-1})$ is quasilinear, we show that ϕ_t is also Gaussian.*

Proof Let ϕ_{t-1} denote arbitrageur's belief on (x_{t-1}, k) at time $t-1$. Let ϕ_{t-1}^+ arbitrageur's belief on (x_{t-1}, k) at time t, after the price p_t is observed. Through the following belief update process, we can recursively update the Gaussian parameters to obtain posterior belief on (x_{t-1}, k). The mean of ϕ_{t-1}^+ is calculated using the following equations:

$$\mu^{+}_{x,t-1} = \theta^t_x \left(\hat{\alpha_{4,t}}\mu_{k,t-1} + \hat{\alpha_{3,t}}\mu_{x,t-1} + \hat{\alpha_{2,t}}y_{t-1}\right) + \theta^t_x \left(\psi_t - \Delta p_t/\lambda\right) + \frac{{\sigma_\epsilon}^2 \left(\sigma_{x,t-1}\mu_{k,t-1}\rho_{x,k,t-1} - \sigma_{k,t-1}\mu_{x,t-1}\right)}{C^t_x}$$

$$\mu^{+}_{k,t-1} = \theta^t_k \left(\hat{\alpha_{4,t}}\mu_{k,t-1} + \hat{\alpha_{3,t}}\mu_{x,t-1} + \hat{\alpha_{2,t}}y_{t-1}\right) + \theta^t_k \left(\psi_t - \Delta p_t/\lambda\right)$$

$$+ \frac{{\sigma_\epsilon}^2 \left(\sigma_{k,t-1}\mu_{x,t-1}\rho_{x,k,t-1} - \sigma_{x,t-1}\mu_{k,t-1}\right)}{C^t_k}$$

The updated covariance matrix has the form,

$$\Sigma^{+}_{t-1} = \begin{bmatrix} \sigma^{+2}_{x,t-1} & \rho^{+}_{x,k,t-1}\sigma^{+}_{x,t-1}\sigma^{+}_{k,t-1} \\ \rho^{+}_{x,k,t-1}\sigma^{+}_{x,t-1}\sigma^{+}_{k,t-1} & \sigma^{+2}_{k,t-1} \end{bmatrix}$$

and is updated using the following equations:

$$-\frac{1}{2\left(1-\rho^{+2}_{x,k,t-1}\right)\sigma^{+2}_{x,t-1}} = m1$$

$$-\frac{1}{2\left(1-\rho^{+2}_{x,k,t-1}\right)\sigma^{+2}_{k,t-1}} = m2$$

$$\frac{\rho^{+}_{x,k,t-1}}{\sigma^{+}_{k,t-1}\sigma^{+}_{x,t-1}\left(1-\rho^{+2}_{x,k,t-1}\right)} + \frac{\lambda^2\hat{\alpha_{1,t}}\hat{\alpha_{5,t}}}{{\sigma_\epsilon}^2} = m3$$

The variables m_1, m_2 and m_3 are defined by the following formulae:

$$m_1 == \frac{C^t_x}{2\sigma_{k,t-1}\sigma^2_{x,t-1}\left(1-\rho^2_{x,k,t-1}\right){\sigma_\epsilon}^2}$$

$$m_2 = \frac{C^t_k}{2\sigma_{x,t-1}\sigma^2_{k,t-1}\left(1-\rho^2_{x,k,t-1}\right){\sigma_\epsilon}^2}$$

$$m_3 = \frac{\rho_{x,k,t-1}}{\sigma_{k,t-1}\sigma_{x,t-1}\left(1-\rho^2_{x,k,t-1}\right)} + \frac{\lambda^2\hat{\alpha_{1,t}}\hat{\alpha_{5,t}}}{{\sigma_\epsilon}^2}$$

Note that,

$$\begin{bmatrix} x_t \\ k_t \end{bmatrix} = \begin{bmatrix} 1+\hat{\alpha_{1,t}} & \hat{\alpha_{5,t}} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x^{+}_{t-1} \\ k^{+}_{t-1} \end{bmatrix} + \begin{bmatrix} \hat{\alpha_{2,t}}y_{t-1} + \hat{\alpha_{3,t}}\mu_{x,t-1} + \hat{\alpha_{4,t}}\mu_{k,t-1} \\ 0 \end{bmatrix}$$

Thus, ϕ_t is also a two-dimensional Gaussian with variables defined as follows:

$$\mu_{x,t} = (1+\hat{\alpha_{1,t}})\mu^{+}_{x,t-1} + \hat{\alpha_{5,t}}\mu^{+}_{k,t-1} + \hat{\alpha_{2,t}}y_{t-1} + \hat{\alpha_{3,t}}\mu_{x,t-1} + \hat{\alpha_{4,t}}\mu_{k,t-1}$$

$$\mu_{k,t} = \mu^{+}_{k,t-1}$$

$$\Sigma_t = \begin{bmatrix} 1+\hat{\alpha_{1,t}} & 0 \\ \hat{\alpha_{5,t}} & 1 \end{bmatrix} \Sigma^{+}_{t-1} \begin{bmatrix} 1+\hat{\alpha_{1,t}} & \hat{\alpha_{5,t}} \\ 0 & 1 \end{bmatrix}$$

Next, we calculate a set of expressions for the expected future mean of the belief given the current information set. These expressions are useful in calculating the value functions recursively, and will be used in §4.3 and §4.4.

Lemma 4.2 *Assume the arbitrageur's estimate of the trader's policy is*

$$\hat{\pi}_t(x_{t-1}, y_{t-1}, \phi_{t-1}) = \hat{\alpha_{1,t}}x_{t-1} + \hat{\alpha_{2,t}}y_{t-1} + \hat{\alpha_{3,t}}\mu_{x,t-1} + \hat{\alpha_{4,t}}\mu_{k,t-1} + \hat{\alpha_{5,t}}k.$$

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We can calculate the following expressions as follows:

$$\mathop{\mathbf{E}}_{u_t}\left[\mu^{+}_{x,t-1}|x_{t-1}, y_{t-1}, \phi_{t-1}\right] = \theta^t_x\left(-u_t + \alpha_{4,t}\mu_{k,t-1} + \alpha_{3,t}\mu_{x,t-1} + \alpha_{2,t}y_{t-1}\right) + \frac{{\sigma_\epsilon}^2\left(\sigma_{x,t-1}\mu_{k,t-1}\rho_{x,k,t-1} - \sigma_{k,t-1}\mu_{x,t-1}\right)}{C^t_x}$$

$$\mathop{\mathbf{E}}_{u_t}\left[\mu^{+}_{k,t-1}|x_{t-1}, y_{t-1}, \phi_{t-1}\right] = \theta^t_k\left(-u_t + \alpha_{4,t}\mu_{k,t-1} + \alpha_{3,t}\mu_{x,t-1} + \alpha_{2,t}y_{t-1}\right) + \frac{{\sigma_\epsilon}^2\left(\sigma_{k,t-1}\mu_{x,t-1}\rho_{x,k,t-1} - \sigma_{x,t-1}\mu_{k,t-1}\right)}{C^t_k}$$

$$\mathop{\mathbf{E}}_{u_t}\left[\mu^{+2}_{x,t-1}|x_{t-1}, y_{t-1}, \phi_{t-1}\right] = \frac{{\theta^t_x}^2{\sigma_\epsilon}^2}{\lambda^2} + \left(\theta^t_x\left(-u_t + \alpha_{4,t}\mu_{k,t-1} + \alpha_{3,t}\mu_{x,t-1} + \alpha_{2,t}y_{t-1}\right) + \frac{{\sigma_\epsilon}^2\left(\sigma_{x,t-1}\mu_{k,t-1}\rho_{x,k,t-1} - \sigma_{k,t-1}\mu_{x,t-1}\right)}{C^t_x}\right)^2$$

$$\mathop{\mathbf{E}}_{u_t}\left[\mu^{+2}_{k,t-1}|x_{t-1}, y_{t-1}, \phi_{t-1}\right] = \frac{{\theta^t_k}^2{\sigma_\epsilon}^2}{\lambda^2} + \left(\theta^t_k\left(-u_t + \alpha_{4,t}\mu_{k,t-1} + \alpha_{3,t}\mu_{x,t-1} + \alpha_{2,t}y_{t-1}\right) + \frac{{\sigma_\epsilon}^2\left(\sigma_{k,t-1}\mu_{x,t-1}\rho_{x,k,t-1} - \sigma_{x,t-1}\mu_{k,t-1}\right)}{C_k}\right)^2$$

$$\mathop{\mathbf{E}}_{u_t}\left[\mu^+_{x,t-1}\mu^+_{k,t-1}|x_{t-1},y_{t-1},\phi_{t-1}\right]=\frac{\theta^t_x\theta^t_k{\sigma_\epsilon}^2}{\lambda^2}$$
$$+\left(\theta^t_x\left(-u_t+\alpha_{4,t}\mu_{k,t-1}+\alpha_{3,t}\mu_{x,t-1}+\alpha_{2,t}y_{t-1}\right)+\frac{{\sigma_\epsilon}^2\left(\sigma_{x,t-1}\mu_{k,t-1}\rho_{x,k,t-1}-\sigma_{k,t-1}\mu_{x,t-1}\right)}{C^t_x}\right)$$
$$\cdot\left(\theta^t_k\left(-u_t+\alpha_{4,t}\mu_{k,t-1}+\alpha_{3,t}\mu_{x,t-1}+\alpha_{2,t}y_{t-1}\right)+\frac{{\sigma_\epsilon}^2\left(\sigma_{k,t-1}\mu_{x,t-1}\rho_{x,k,t-1}-\sigma_{x,t-1}\mu_{k,t-1}\right)}{C^t_k}\right)$$

$$\mathop{\mathbf{E}}_{u_t}\left[\mu_{x,t}|x_{t-1},y_{t-1},\phi_{t-1}\right]=\left(1+\alpha_{1,t}\right)\mathop{\mathbf{E}}_{u_t}\left[\mu^+_{x,t-1}|x_{t-1},y_{t-1},\phi_{t-1}\right]+\alpha_{5,t}\mathop{\mathbf{E}}_{u_t}\left[\mu^+_{k,t-1}|x_{t-1},y_{t-1},\phi_{t-1}\right]$$
$$+\alpha_{2,t}y_{t-1}+\alpha_{3,t}\mu_{x,t-1}+\alpha_{4,t}\mu_{k,t-1}$$

$$\mathop{\mathbf{E}}_{u_t}\left[\mu_{k,t}|x_{t-1},y_{t-1},\phi_{t-1}\right]=\mathop{\mathbf{E}}_{u_t}\left[\mu^+_{k,t-1}|x_{t-1},y_{t-1},\phi_{t-1}\right]$$

$$\mathop{\mathbf{E}}_{u_t}\left[\mu^2_{x,t}|x_{t-1},y_{t-1},\phi_{t-1}\right]=\left(1+\alpha_{1,t}\right)^2\mathop{\mathbf{E}}_{u_t}\left[\mu^{+2}_{x,t-1}|x^{+2}_{t-1},y_{t-1},\phi_{t-1}\right]+\alpha_5^2\mathop{\mathbf{E}}_{u_t}\left[\mu^{+2}_{k,t-1}|x_{t-1},y_{t-1},\phi_{t-1}\right]+\alpha^2_{2,t}y^2_{t-1}$$
$$+\alpha^2_{3,t}\mu^2_{x,t-1}+\alpha_{4,t}\mu^2_{k,t-1}+2\left(1+\alpha_{1,t}\right)\alpha_{5,t}\mathop{\mathbf{E}}_{u_t}\left[\mu^+_{x,t-1}\mu^+_{k,t-1}|x_{t-1},y_{t-1},\phi_{t-1}\right]$$
$$+2\left(1+\alpha_{1,t}\right),\left(\alpha_{2,t}y_{t-1}+\alpha_{3,t}\mu_{x,t-1}+\alpha_{4,t}\mu_{k,t-1}\right)\mathop{\mathbf{E}}_{u_t}\left[\mu^+_{x,t-1}|x_{t-1},y_{t-1},\phi_{t-1}\right]$$
$$+\left(2\,\alpha_{5,t}\alpha_2y_{t-1}+2\,\alpha_{5,t}\alpha_{3,t}\mu_{x,t-1}+2\,\alpha_{5,t}\alpha_{4,t}\mu_{k,t-1}\right)\mathop{\mathbf{E}}_{u_t}\left[\mu^+_{k,t-1}|x_{t-1},y_{t-1},\phi_{t-1}\right]$$
$$+2\,\alpha_{2,t}\alpha_{3,t}y_{t-1}\mu_{x,t-1}+2\,\alpha_{2,t}\alpha_{4,t}y_{t-1}\mu_{k,t-1}+2\,\alpha_{3,t}\alpha_{4,t}\mu_{k,t-1}\mu_{x,t-1}$$

$$\mathop{\mathbf{E}}_{u_t}\left[\mu^2_{k,t}|x_{t-1},y_{t-1},\phi_{t-1}\right]=\mathop{\mathbf{E}}_{u_t}\left[\mu^{+2}_{k,t-1}|x_{t-1},y_{t-1},\phi_{t-1}\right]$$

$$\mathop{\mathbf{E}}_{u_t}\left[\mu_{x,t}\mu_{k,t}|x_{t-1},y_{t-1},\phi_{t-1}\right]=\left(1+\alpha_{1,t}\right)\mathop{\mathbf{E}}_{u_t}\left[\mu^+_{x,t-1}\mu^+_{k,t-1}|x_{t-1},y_{t-1},\phi_{t-1}\right]+\alpha_{5,t}\mathop{\mathbf{E}}_{u_t}\left[\mu^{+2}_{k,t-1}|x_{t-1},y_{t-1},\phi_{t-1}\right]$$
$$+\alpha_{2,t}y_{t-1}+\alpha_{3,t}\mu_{x,t-1}+\alpha_{4,t}\mu_{k,t-1}\mathop{\mathbf{E}}_{u_t}\left[\mu^+_{k,t-1}|x_{t-1},y_{t-1},\phi_{t-1}\right]$$

Similarly,

$$\mathop{\mathbf{E}}_{v_t}\left[\mu^+_{x,t-1}|y_{t-1},\phi_{t-1}\right]=\theta_x\left(-\alpha_{1,t}\mu_{x,t-1}-\alpha_{5,t}\mu_{k,t-1}\right)+\frac{\sigma^2_\epsilon\left(\sigma_{x,t-1}\mu_{k,t-1}\rho_{x,k,t-1}-\sigma_{k,t-1}\mu_{x,t-1}\right)}{C_x}$$

$$\mathop{\mathbf{E}}_{v_t}\left[\mu^+_{k,t-1}|y_{t-1},\phi_{t-1}\right]=\theta_k\left(-\alpha_{1,t}\mu_{x,t-1}-\alpha_{5,t}\mu_{k,t-1}\right)+\frac{\sigma^2_\epsilon\left(\sigma_{k,t-1}\mu_{x,t-1}\rho_{x,k,t-1}-\sigma_{x,t-1}\mu_{k,t-1}\right)}{C_k}$$

$$\mathop{\mathbf{E}}_{v_t}\left[\mu_{x,t-1}^{+2}|y_{t-1},\phi_{t-1}\right]=\frac{\theta_x^2{\sigma_\epsilon}^2}{\lambda^2}+\left(\theta_x^t\left(-\alpha_{1,t}\mu_{x,t-1}-\alpha_{5,t}\mu_{k,t-1}\right)+\frac{\sigma_\epsilon^2\left(\sigma_{x,t-1}\mu_{k,t-1}\rho_{x,k,t-1}-\sigma_{k,t-1}\mu_{x,t-1}\right)}{C_x}\right)^2$$

$$\mathop{\mathbf{E}}_{v_t}\left[\mu_{k,t-1}^{+2}|y_{t-1},\phi_{t-1}\right]=\frac{\theta_k^2{\sigma_\epsilon}^2}{\lambda^2}+\left(\theta_k^t\left(-\alpha_{1,t}\mu_{x,t-1}-\alpha_{5,t}\mu_{k,t-1}\right)+\frac{\sigma_\epsilon^2\left(\sigma_{k,t-1}\mu_{x,t-1}\rho_{x,k,t-1}-\sigma_{x,t-1}\mu_{k,t-1}\right)}{C_k}\right)^2$$

$$\begin{aligned}\mathop{\mathbf{E}}_{v_t}\left[\mu_{x,t-1}^{+}\mu_{k,t-1}^{+}|y_{t-1},\phi_{t-1}\right]=&\frac{\theta_x\theta_k{\sigma_\epsilon}^2}{\lambda^2}\\&+\left(\theta_x\left(-\alpha_{1,t}\mu_{x,t-1}-\alpha_{5,t}\mu_{k,t-1}\right)+\frac{{\sigma_\epsilon}^2\left(\sigma_{x,t-1}\mu_{k,t-1}\rho_{x,k,t-1}-\sigma_{k,t-1}\mu_{x,t-1}\right)}{C_x}\right)\\&.\left(\theta_k\left(-\alpha_{1,t}\mu_{x,t-1}-\alpha_{5,t}\mu_{k,t-1}\right)+\frac{{\sigma_\epsilon}^2\left(\sigma_{k,t-1}\mu_{x,t-1}\rho_{x,k,t-1}-\sigma_{x,t-1}\mu_{k,t-1}\right)}{C_k}\right)\end{aligned}$$

$$\begin{aligned}\mathop{\mathbf{E}}_{v_t}\left[\mu_{x,t}|y_{t-1},\phi_{t-1}\right]=&(1+\alpha_{1,t})\mathop{\mathbf{E}}_{v_t}\left[\mu_{x,t-1}^{+}|y_{t-1},\phi_{t-1}\right]+\alpha_{5,t}\mathop{\mathbf{E}}_{v_t}\left[\mu_{k,t-1}^{+}|y_{t-1},\phi_{t-1}\right]+\alpha_{2,t}y_{t-1}+\alpha_{3,t}\mu_{x,t-1}\\&+\alpha_{4,t}\mu_{k,t-1}\end{aligned}$$

$$\mathop{\mathbf{E}}_{v_t}\left[\mu_{k,t}|y_{t-1},\phi_{t-1}\right]=\mathop{\mathbf{E}}_{v_t}\left[\mu_{k,t-1}^{+}|y_{t-1},\phi_{t-1}\right]$$

$$\begin{aligned}\mathop{\mathbf{E}}_{v_t}\left[\mu_{x,t}^2|y_{t-1},\phi_{t-1}\right]=&(1+\alpha_{1,t})^2\mathop{\mathbf{E}}_{v_t}\left[\mu_{x,t-1}^{+2}|y_{t-1},\phi_{t-1}\right]+\alpha_5^2\mathop{\mathbf{E}}_{v_t}\left[\mu_{k,t-1}^{+2}|y_{t-1},\phi_{t-1}\right]+\alpha_2^2y_{t-1}^2+\alpha_{3,t}^2\mu_{x,t-1}^2\\&+\alpha_{4,t}\mu_{k,t-1}^2+2\ (1+\alpha_1)\,\alpha_{5,t}\mathop{\mathbf{E}}_{v_t}\left[\mu_{x,t-1}^{+}\mu_{k,t-1}^{+}|y_{t-1},\phi_{t-1}\right]\\&+2\,(1+\alpha_1)\,(\alpha_{2,t}y_{t-1}+\alpha_{3,t}\mu_{x,t-1}+\alpha_{4,t}\mu_{k,t-1})\mathop{\mathbf{E}}_{v_t}\left[\mu_{x,t-1}^{+}|y_{t-1},\phi_{t-1}\right]\\&+(2\,\alpha_{5,t}\alpha_2y_{t-1}+2\,\alpha_{5,t}\alpha_{3,t}\mu_{x,t-1}+2\,\alpha_{5,t}\alpha_{4,t}\mu_{k,t-1})\mathop{\mathbf{E}}_{v_t}\left[\mu_{k,t-1}^{+}|y_{t-1},\phi_{t-1}\right]\\&+2\,\alpha_{2,t}\alpha_{3,t}y_{t-1}\mu_{x,t-1}+2\,\alpha_{2,t}\alpha_{4,t}y_{t-1}\mu_{k,t-1}+2\,\alpha_{3,t}\alpha_{4,t}\mu_{k,t-1}\mu_{x,t-1}\end{aligned}$$

$$\mathop{\mathbf{E}}_{v_t}\left[\mu_{k,t}^2|y_{t-1},\phi_{t-1}\right]=\mathop{\mathbf{E}}_{v_t}\left[\mu_{k,t-1}^{+2}|y_{t-1},\phi_{t-1}\right]$$

$$\begin{aligned}\mathop{\mathbf{E}}_{v_t}\left[\mu_{x,t}\mu_{k,t}|y_{t-1},\phi_{t-1}\right]=&(1+\alpha_{1,t})\mathop{\mathbf{E}}_{v_t}\left[\mu_{x,t-1}^{+}\mu_{k,t-1}^{+}|y_{t-1},\phi_{t-1}\right]+\alpha_{5,t}(\mathop{\mathbf{E}}_{v_t}\left[\mu_{k,t-1}^{+2}|y_{t-1},\phi_{t-1}\right])\\&+\alpha_{2,t}y_{t-1}+\alpha_{3,t}\mu_{x,t-1}+\alpha_{4,t}\mu_{k,t-1}\mathop{\mathbf{E}}_{v_t}\left[\mu_{k,t-1}^{+}|y_{t-1},\phi_{t-1}\right]\end{aligned}$$

4.3 Trader's Value Function

At each time, the trader's objective is to maximize the expected future profit. The expected future profit at $t = T-1$ can be initialized to

$$U^*_{T-1}(x_{T-1}, y_{T-1}, \phi_{T-1}) = \mathbf{E}[x_{T-1}(kp_0 - p_{T-1})] = x_{T-1}(p_0(k-1) - \lambda y_{T-1}) - \lambda x^2_{T-1}$$

At time $t = T-1, \ldots, 2$, we can calculate the value function using backward recursion,

$$U^{\pi,(\psi,\hat{\pi})}_{t-1}(x_{t-1}, y_{t-1}, \phi_{t-1}) = \max_{u_t} \mathbf{E}[x_{t-1}\Delta p_t + U^*_t(x_t, y_t, \phi_t)|x_{t-1}, y_{t-1}, \phi_{t-1}]$$

.

Theorem 4.3 *Assuming that the arbitrageur uses a quasi-linear policy denoted by,*

$$v_t = \psi_t(y_{t-1}, \phi_{t-1}) = \beta_{1,t} y_{t-1} + \beta_{2,t}\mu_{x,t-1} + \beta_{3,t}\mu_{k,t-1}$$

,

and arbitrageur's estimate of trader's action is quasi-linear and is denoted by,

$$u_t = \hat{\pi}_t(x_{t-1}, y_{t-1}, \phi_{t-1}) = \alpha_{1,t}x_{t-1} + \alpha_{2,t}y_{t-1} + \alpha_{3,t}\mu_{x,t-1} + \alpha_{4,t}\mu_{k,t-1} + \alpha_{5,t}k$$

,

the value function takes the following form:

$$\begin{aligned} U^*_{t-1}(x_{t-1}, y_{t-1}, \phi_{t-1}) = {} & c_1 x^2_{t-1} + c_2 y^2_{t-1} + c_3\mu^2_{x,t-1} + c_4\mu^2_{k,t-1} + c_5 k^2 + c_6 x_{t-1}y_{t-1} + c_7 x_{t-1}\mu_{x,t-1} \\ & + c_8 x_{t-1}\mu_{k,t-1} + c_9 x_{t-1}k + c_{10}y_{t-1}\mu_{x,t-1} + c_{11}y_{t-1}\mu_{k,t-1} + c_{12}y_{t-1}k \\ & + c_{13}\mu_{x,t-1}\mu_{k,t-1} + c_{14}\mu_{x,t-1}k + c_{15}\mu_{k,t-1}k + c_{16}. \end{aligned} \quad (1)$$

Proof Note that,

$$U^*_{T-1}(x_{T-1}, y_{T-1}, \phi_{T-1}) = x_{T-1}(p_0(k-1) - \lambda y_{T-1}) - \lambda x^2_{T-1}$$

,

Thus, $c_1(T-1) = -\lambda$, $c_6(T-1) = -\lambda$, $c_9(T-1) = p_0$, and $c_i(T-1) = 0$ otherwise. Assume that the theorem holds for t. Then,

$$\begin{aligned} U^{\pi,(\psi,\hat{\pi})}_{t-1}(x_{t-1}, y_{t-1}, \phi_{t-1}) &= \max_{u_t} \mathbf{E}[x_{t-1}\Delta p_t + U^*_t(x_t, y_t, \phi_t)|x_{t-1}, y_{t-1}, \phi_{t-1}] \\ &= \max_{u_t} \mathbf{E}[\lambda x_{t-1}(u_t + v_t + \epsilon_t/\lambda) + U^*_t(x_t, y_t, \phi_t)|x_{t-1}, y_{t-1}, \phi_{t-1}] \\ &= \max_{u_t} \mathbf{E}[\lambda x_{t-1}(u_t + \beta_{1,t}y_{t-1} + \beta_{2,t}\mu_{x,t-1} + \beta_{3,t}\mu_{k,t-1} + \epsilon_t/\lambda) \\ &+ U^*_t(x_{t-1} + u_t, y_{t-1}(1+\beta_{1,t}) + \beta_{2,t}\mu_{x,t-1} + \beta_{3,t}\mu_{k,t-1}, \phi_t)|x_{t-1}, y_{t-1}, \phi_{t-1}] \\ &= \max_{u_t}[\lambda x_{t-1}(u_t + \beta_{1,t}y_{t-1} + \beta_{2,t}\mu_{x,t-1} + \beta_{3,t}\mu_{k,t-1}) \\ &+ \mathbf{E}[U^*_t(x_{t-1} + u_t, y_{t-1}(1+\beta_{1,t}) + \beta_{2,t}\mu_{x,t-1} + \beta_{3,t}\mu_{k,t-1}, \phi_t)|x_{t-1}, y_{t-1}, \phi_{t-1}]] \end{aligned}$$

Using Lemma 4.2, the mean and variance of ϕ_t can be calculated as a function of the information set at $t-1$. Thus, U_t has a quadratic form, which is minimized by a quasi-linear choice of u_t. The resulting U^*_{t-1} has the same quadratic form as required by the theorem. The coefficients of U^*_{t-1} can be calculated using a backward recursion.

4.4 Arbitrageur's Value Function

Assume that the arbitrageur believes that the trader uses a policy $\hat{\pi}_t$ at time t of the following form:

$$\pi_t(x_{t-1}, y_{t-1}, \phi_{t-1}) = \alpha_{1,t}x_{t-1} + \alpha_{2,t}y_{t-1} + \alpha_{3,t}\mu_{x,t-1} + \alpha_{4,t}\mu_{k,t-1} + \alpha_{5,t}kp_0$$

Theorem 4.4 *The arbitrageur's value function has the form,*

$$V^*_t(y_t, \phi_t) = d_1 y_t^2 + d_2\mu^2_{x,t-1} + d_3\mu^2_{k,t-1} + d_4 y_{t-1}\mu_{x,t-1} + d_5 y_{t-1}\mu_{k,t-1} + d_6\mu_{x,t-1}\mu_{k,t-1} + d_7$$

Proof The expected future profit at $t = T-1$ can be initialized to

$$V^*_{T-1}(x_{T-1}, y_{T-1}, \phi_{T-1}) = \lambda(y_{T-1} + \mu_{T-1})y_{T-1} \tag{2}$$

Thus, $d_1(T-1) = \lambda$, $d_4(T-1) = \lambda$, and $d_i(T-1) = 0$ otherwise.

Assume that the theorem holds for t. Then,

$$\begin{aligned}
V_{t-1}(y_{t-1}, \phi_{t-1}) &= \max_{v_t} \mathbf{E}[y_{t-1}\Delta p_t + V^*_t(y_t, \phi_t)|y_{t-1}, \phi_{t-1}] \\
&= \max_{v_t} \mathbf{E}[\lambda y_{t-1}(u_t + v_t + \epsilon_t/\lambda) + V^*_t(y_t, \phi_t)|y_{t-1}, \phi_{t-1}] \\
&= \max_{v_t} \mathbf{E}[\lambda y_{t-1}(\alpha_{1,t}x_{t-1} + \alpha_{2,t}y_{t-1} + \alpha_{3,t}\mu_{x,t-1} + \alpha_{4,t}\mu_{k,t-1} + \alpha_{5,t}kp_0 + v_t) \\
&+ V^*_t(y_{t-1}((1+\alpha_{1,t})x_{t-1} + \alpha_{2,t}y_{t-1} + \alpha_{3,t}\mu_{x,t-1} + \alpha_{4,t}\mu_{k,t-1} + \alpha_{5,t}kp_0, \phi_t)|y_{t-1}, \phi_{t-1}] \\
&= \max_{v_t}[\lambda y_{t-1}(\alpha_{1,t}x_{t-1} + \alpha_{2,t}y_{t-1} + \alpha_{3,t}\mu_{x,t-1} + \alpha_{4,t}\mu_{k,t-1} + \alpha_{5,t}kp_0 + v_t) \\
&+ \mathbf{E}[V^*_t(y_{t-1}((1+\alpha_{1,t})x_{t-1} + \alpha_{2,t}y_{t-1} + \alpha_{3,t}\mu_{x,t-1} + \alpha_{4,t}\mu_{k,t-1} + \alpha_{5,t}kp_0, \phi_t)|y_{t-1}, \phi_{t-1}]]
\end{aligned}$$

Using Lemma 4.2, the mean and variance of ϕ_t can be calculated as a function of the information set at $t-1$. Thus, V_t has a quadratic form, which is minimized by a quasi-linear choice of v_t. The resulting V^*_{t-1} has the same quadratic form as required by the theorem. The coefficients of V^*_{t-1} can be calculated using a backward recursion.

4.5 Optimal Strategies

In each of the previous two sections, §4.3 and §4.4, we are assuming the policy of one player and derive the optimal policy of the other. Since these policies coincide, it turns out that the adopted policies result in a fixed point in the game.

The resulting policies have the form, $u_t = \alpha_{1,t}x_{t-1} + \alpha_{2,t}y_{t-1} + \alpha_{3,t}\mu_{x,t-1} + \alpha_{4,t}k_{t-1} + \alpha_{5,t}k$, $v_t = \beta_{1,t}y_{t-1} + \beta_{2,t}\mu_{x,t-1} + \beta_{3,t}\mu_{k,t-1}$, where $\alpha_{1,t}$, $\alpha_{2,t}$, $\alpha_{3,t}$, $\alpha_{4,t}$, $\alpha_{5,t}$, $\beta_{1,t}$, $\beta_{2,t}$, $\beta_{3,t}$ are updated recursively.

At the PBE, $\pi^* = \hat{\pi}$; thus, we can calculate α_t and β_t from two 7^{th} order polynomial systems. The system of equations for solving α_t consists of five equations: two of them have order 7, and the rest are coupled to to the first two, and are first order. The system of equations for solving β_t is first order and coupled to the seventh-order equations of the α_t system of equations. Theoretically, this results in 49 possible solutions for u_t and v_t, some of which will be discarded as unacceptable solutions.

4.6 Calculating the Value Functions using Dynamic Programming

The value functions at time $T-1$ are known. We can thus calculate the value functions using dynamic programming, and calculate the optimal strategy and the coefficients of the new value function recursively.

5 Remaining Work

The next step is to devise a computational method for solving the problem. The following steps form an outline for the method:

1. Initialize U^*_{T-1} and V^*_{T-1} for $(x_{T-1}, y_{T-1}, \phi_{T-1})$
2. Initialize $u_T = -x_{T-1}$ and $v_T = -y_{T-1}$.
3. For $t = T-1, \ldots, 1$
   -Recursively calculate U_{t-1} and V_{t-1}
   -Calculate u_t andv_t by solving for α_t and β_t. Theoretically, since each system of equations consists of two 7^{th} order equations and 5 first order equations, this results in 49 solutions. Appropriate boundary conditions should be established in order to determine the acceptable solution from this set.

After finding the optimal trading strategy for the trader and the arbitrageur, the next interesting question would be look at how the price changes due to these actions, and to make observations about its pattern. In [Kyl85], it is observed that in the continuous time limit, prices follow a brownian motion. It would be interesting to see how this would be affected by adopting the optimal strategy proposed in this report.

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